A partial Holder regularity result for harmonic maps into a Finsler manifold (Variational Problems and Related Topics)

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A partial Hölder regularity result for harmonic maps into a Finsler manifold

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1 Introduction

It goes without saying that many theories on differential equations contribute to the investigation of the differential geometric problems. Inversely,we can say that some important geometric problems give strong motivation to investigate corresponding analytic problems and contributes to developments of (nonlinear) analysis. As a typical example, we mention *harmonic maps.* Since the pioneering work by J.Eells and J.H.Sampson [5] was published, *harmonic maps between Riemannian manifolds* have attracted great interests of many nonlinear analysts as well as geometers. Especially, regularity problems for *weakly harmonic maps* have been one of the most attractive problems for researchers of quadratic variational problems. Nowadays, we have many detailed results on regularity or/and singularity of solutions of variational problems for the following type of functionals in relation to harmonic maps:

$$\int a_{\alpha\beta}(x)b^{ij}(u)\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}dx.$$

Harmonic maps between Riemannian manifolds are defined as follows. Let (M,g) and (N,h) be Riemannian $m-$ and $n-$manifolds respectively. *The energy density* of a map $u:(M,g)\to(N,h)$ is the function $e(u):M\to\mathbb{R}$ defined by

$$e(u)(p)=\frac{1}{2}|du(p)|^2,$$

where $|du|$ denotes the Hilbert-Schmidt norm of $du(p)\in T_p^*M\otimes T_{u(p)}N$, namely for a orthonormal basis $(e_1,...,e_m)$ of T_pM $e(u)(p)$ can be written as

$$e(u)(p)=\sum_{\alpha=1}^{m}\|(du(p))(e_\alpha)\|^2_{T_{u(p)}N}. \tag{1.1}$$

For a bounded domain Ω in M, *the energy* of u on Ω is defined by

$$E(u,\Omega)=\int_\Omega e(u)d\mu,$$

where $d\mu$ stands for the volume element on M. *Harmonic maps* are defined as solutions of the Euler-Lagrange equations of the energy functional.

Since Finsler geometry is a natural generalization of Riemannian geometry, from the viewpoint of differential geometry, it is very natural to expect to extend the important notion of harmonicity to Finsler geometry. In fact, P.Centore [4] defined the energy of a map between Finsler manifolds and introduced a new notion of harmonicity of maps between Finsler manifolds. On the other hand, X.Mo [15] defined harmonic maps from Finsler manifolds into Riemannian manifolds independently and got interesting results.

It will be worth to consider Finsler geometric variational problems from the viewpoint of applied mathematics as well. If we can consider a given variational problem in a geometric context, sometimes we will get good suggestions and even fine perspective for the problem. On the other hand, there are many variational and evolution problems which can not be interpreted as Riemannian geometric problems, and some of them can be regarded as geometric problems in Finsler geometry. For example, many physical and biological applications of Finsler geometry are introduced by P.L.Antonelli, R.S.Ingarten and M.Matsumoto in [1]. We mention also that G.Bellettini and M.Paolini [3] studied a problem of anisotropic motion in a context of Finsler geometry.

As in the Riemannian case, the regularity problem for harmonic maps between Finsler manifolds should be very interesting problem.

Now, we give the definition of Finsler manifold in accordance with [2] and introduce the definition of *the energy density of a map between Finsler manifolds* defined in [4].

Let N be a n-dimensional C^∞ manifold and TN the tangent bundle of N. We express each point in TN as (u, X) with $u \in N$ and $X \in T_xN$. Moreover, $\mathbf{0}$ denotes the $0-$section $\{(u,0)\} \subset TN$ when we write $TN \setminus \mathbf{0}$. *A Finsler structure* of N is a function

$$\begin{array}{cccc} F: & TN & \longrightarrow & [0,\infty) \\ & \Psi & & \Psi \\ & (u,X) & \longmapsto & F(u,X) \end{array}$$

with the following properties:

(i) Regularity: $F \in C^\infty(TN \setminus \mathbf{0})$.

(ii) Homogeneity:

$$F(u, \lambda X) = \lambda F(u, X) \quad \text{for all } \lambda \geq 0. \tag{1.2}$$

(iii) Convexity: The Hessian matrix of F^2 with respect to X

$$(h_{ij}(u,X)) = \left(\frac{1}{2} \frac{\partial^2 F^2(u,X)}{\partial X^i \partial X^j} \right)$$

is positive definite at every point $(u, X) \in TN \setminus \mathbf{0}$.

We call the pair (N, F) *a Finsler manifold.*

Centore [4] defines the energy density of maps between Finsler manifold as follows.

Definition 1.1 (Centore). *Let (M,G) and (N,F) be Finsler manifolds, and I_xM a unit ball in T_xM, namely*

$$I_xM = \{\xi \in T_xM; G(X) \leq 1\}.$$

For a C^1-map $u : M \to N$, we define the energy density $e(u)(x)$ of u at $x \in M$ by

$$e(u)(x) = \frac{\int_{I_xM}(u^*F)^2(X)dX}{\int_{I_xM} dX}. \tag{1.3}$$

Centore showed that the above definition is consistent with the standard one (1.1) for Riemannian cases ([4, Lemma 2]) and also that it can be regarded as a special case of the definition given by Jost [13] for abstract cases of metric measure spaces ([4, Theorem 5]).

In this paper, we consider harmonic maps from $\mathbb{R}^m$ into Finsler space $(N,F) = (\mathbb{R}^n, F)$. For such a case, by virtue of the special structure of $\mathbb{R}^m$ and the homogeneity of F, we can write the energy density defined by (1.3) more simply. Namely, for a map $u : \mathbb{R}^m \to (N,F)$ we can define the energy density of u at $x \in \mathbb{R}^m$ as

$$e(u)(x) = \frac{1}{2c_m}\int_{S^{m-1}} F^2(u(x), du_x \cdot \xi)d\mathcal{H}^{m-1}(\xi), \tag{1.4}$$

where $\mathcal{H}^{m-1}$ denotes the $(m-1)$-dimensional Hausdorff measure, $c_m = \mathcal{H}^{m-1}(S^{m-1})$, du_x denotes the differential of u at x and

$$\begin{aligned} du_x \cdot \xi &= \left((du_x \cdot \xi)^1, ..., (du_x \cdot \xi)^n\right) \\ &= \left(\frac{\partial u^1}{\partial x^\alpha}\xi^\alpha, ..., \frac{\partial u^n}{\partial x^\alpha}\xi^\alpha\right). \end{aligned}$$

Moreover, as usual, we define *the energy of u on $\Omega \subset \mathbb{R}^m$* by

$$\begin{aligned} E(u,\Omega) &= \int_\Omega e(u)(x)dx \\ &= \int_\Omega \left\{\frac{1}{2c_m}\int_{S^{m-1}} F^2(u(x), du_x \cdot \xi)d\mathcal{H}^{m-1}(\xi)\right\} dx. \end{aligned} \tag{1.5}$$

We consider minimizing problems for the functional $E(u,\Omega)$ defined by (1.5) in the class

$$H_f^{1,2}(\Omega, N) = \{u \in H^{1,2}(\Omega, N) \ ; \ u - f \in H_0^{1,2}(\Omega, \mathbb{R}^n)\}, \tag{1.6}$$

where $f : \partial\Omega \to N$ is a given L^∞ function. In the following, "a minimizer of E" means "a minimizer of E in the class $H_f^{1,2}(\Omega, N)$".

As in the case that the target manifold is Riemannian, let us call a solution of the Euler-Lagrange equation of (1.5) to be *a harmonic map.*

The aim of this article is to show the partial Hölder regularity of energy minimizing map $u : \mathbb{R}^m \to (N,F)$ for the case that $m = 3, 4$.

2 Harmonic Map

For homogeneous functions the following theorem is known as Euler's Theorem (see, for example [2, Theorem 1.2.1]).

Theorem 2.1 (Euler). *Let a function $H : \mathbb{R}^n \to \mathbb{R}$ be differentiable on $\mathbb{R}^n \setminus \{0\}$, then the following two statements are equivalent:*

- *H is positively homogeneous of degree r. That is*

$$H(\lambda X) = \lambda^r H(X) \qquad \text{for all } \lambda > 0 \text{ and } X \in \mathbb{R}^n. \tag{2.1}$$

- *The radial directional derivative of H is equal to rH. Namely*

$$X^i H_{X^i}(X) = rH(X) \qquad \text{for all } X \in \mathbb{R}^n. \tag{2.2}$$

Since $F^2(u, X)$ is positively homogeneous of degree 2 with respect to X, using Euler's theorem, we can easily see that

$$F^2(u, X) = \frac{1}{2} \frac{\partial^2 F^2(u, X)}{\partial X^i \partial X^j} X^i X^j \tag{2.3}$$

Put

$$h_{ij}(u, X) = \frac{1}{4c_m} \frac{\partial^2 f(u, X)}{\partial X^i \partial X^j}. \tag{2.4}$$

Each $h_{ij}(u, X)$ is differentiable in $u \in N$ and $X \in \mathbb{R}^n \setminus \{0\}$ and 0-homogeneous with respect to X. The energy density and the energy of $u : \Omega \subset \mathbb{R}^m \to N$ can be written as follows:

$$e(u)(x) = \frac{1}{2} \int_{S^{m-1}} h_{ij}(u, du_x \cdot \xi) D_\alpha u^i \xi^\alpha D_\beta u^j \xi^\beta d\mathcal{H}^{m-1}(\xi) \tag{2.5}$$

and

$$E(u; \Omega) = \int_\Omega \frac{1}{2} \int_{S^{m-1}} h_{ij}(u, du_x \cdot \xi) D_\alpha u^i \xi^\alpha D_\beta u^j \xi^\beta d\mathcal{H}^{m-1}(\xi) dx \tag{2.6}$$

respectively.

Throughout the paper this article, at most the second derivatives of $F^2(u, X)$ with respect to X appear and the first derivatives of F^2 are Lipschitz continuous. Therefore, by virtue of the chain rule for for compositions of Nemitsky operators and Sobolev maps (see, for example, [14]), every term which contains Du can be defined as zero when $Du = 0$.

We can calculate the first variation of E as

$$
\begin{aligned}
0 = & \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E(u+\varepsilon\phi) \\
= & \int_\Omega \Bigg[\int_{S^{m-1}} \{h_{ij}(u, du\cdot\xi)\xi^\alpha\xi^\beta \\
& \quad + \frac{1}{2} h_{i\ell;j}(u, du\cdot\xi) D_\gamma u^\ell \xi^\gamma \xi^\alpha \xi^\beta\} d\mathcal{H}^{m-1}(\xi) D_\alpha u^i D_\beta \varphi^j \\
& + \frac{1}{2}\int_{S^{m-1}} h_{i\ell,j}(u, du\cdot\xi)\xi^\alpha\xi^\beta d\mathcal{H}^{m-1}(\xi) D_\alpha u^i D_\beta u^\ell \varphi^j \Bigg] dx,
\end{aligned}
\tag{2.7}
$$

for all $\varphi \in C_0^\infty(\Omega)$. Here and in the sequel we write

$$
h_{i\ell,j}(u,X) = \frac{\partial h_{i\ell}}{\partial u^j}, \quad h_{i\ell;j}(u,X) = \frac{\partial h_{i\ell}}{\partial X^j}.
$$

Moreover, since h_{ij} are homogeneous of degree 0, Euler's theorem implies that

$$
0 = \frac{\partial h_{ij}(u,tX)}{\partial t} = h_{ij;\ell}(u,X)X^\ell. \tag{2.8}
$$

On the other hand, we have

$$
h_{ij;\ell}(u,X) = \frac{1}{4c_m}\frac{\partial^3 F^2(u,X)}{\partial X^i \partial X^j \partial X^\ell} = \frac{1}{4c_m}\frac{\partial^3 F^2(u,X)}{\partial X^i \partial X^\ell \partial X^j} = h_{i\ell;j}(u,X). \tag{2.9}
$$

Therefore, the second term in the right-hand side (2.7) vanishes. Thus the weak form of the Euler-Lagrange equation of $E(u;\Omega)$, is given as

$$
\begin{aligned}
0 = & \int_\Omega \Bigg[\int_{S^{m-1}} h_{ij}(u, du\cdot\xi)\xi^\alpha\xi^\beta d\mathcal{H}^{m-1}(\xi) D_\alpha u^i D_\beta \varphi^j \\
& + \frac{1}{2}\int_{S^{m-1}} h_{i\ell,j}(u, du\cdot\xi)\xi^\alpha\xi^\beta d\mathcal{H}^{m-1}(\xi) D_\alpha u^i D_\beta u^\ell \varphi^j \Bigg] dx.
\end{aligned}
\tag{2.10}
$$

If u is of class C^2, by integration by parts, we get that

$$
\begin{aligned}
0 = & \int_\Omega \Bigg[- \int_{S^{m-1}} h_{ij}(u, du\cdot\xi)\xi^\alpha\xi^\beta d\mathcal{H}^{m-1}(\xi) D_\beta D_\alpha u^i \varphi^j \\
& - \int_{S^{m-1}} h_{ij;\ell}(u, du\cdot\xi)\xi^\alpha\xi^\beta\xi^\gamma d\mathcal{H}^{m-1}(\xi) D_\beta D_\gamma u^\ell D_\alpha u^i \varphi^j \\
& - \int_{S^{m-1}} h_{ij,\ell}(u, du\cdot\xi)\xi^\alpha\xi^\beta d\mathcal{H}^{m-1}(\xi) D_\alpha u^i D_\beta u^\ell \varphi^j \\
& + \frac{1}{2}\int_{S^{m-1}} h_{i\ell,j}(u, du\cdot\xi)\xi^\alpha\xi^\beta d\mathcal{H}^{m-1}(\xi) D_\alpha u^i D_\beta u^\ell \varphi^j \Bigg] dx.
\end{aligned}
\tag{2.11}
$$

where
$$ {}^N\Gamma^i_{jk} := h^{is}\frac{1}{2}\left(h_{sj,k} - h_{jk,s} + h_{ks,j}\right). $$
Also in the above equation, the second term vanishes by (2.8) and (2.9). Writing
$$ \gamma^i_{jk} := h^{is}\frac{1}{2}\left(h_{sj,k} - h_{jk,s} + h_{ks,j}\right), \tag{2.12} $$
we obtain from (2.11)
$$ \begin{aligned} &\int_\Omega \Big[-\int_{S^{m-1}} h_{ij}(u, du\cdot\xi)\xi^\alpha\xi^\beta d\mathcal{H}^{m-1}(\xi) D_\beta D_\alpha u^i \varphi^j \\ &\quad - \int_{S^{m-1}} h_{ij}(u, du\cdot\xi)\gamma^i_{k\ell}(u, du\cdot\xi)\xi^\alpha\xi^\beta d\mathcal{H}^{m-1}(\xi) D_\alpha u^k D_\beta u^\ell \varphi^j \Big]\, dx \\ &= 0. \end{aligned} \tag{2.13} $$
Thus we have the Euler-Lagrange equation of the energy:
$$ \begin{cases} \left\{\int_{S^{m-1}} h_{ij}(u, du\cdot\xi)\xi^\alpha\xi^\beta d\mathcal{H}^{m-1}(\xi)\right\} D_\beta D_\alpha u^i \\ + \left\{\int_{S^{m-1}} h_{ij}(u, du\cdot\xi)\gamma^i_{k\ell}(u, du\cdot\xi)\xi^\alpha\xi^\beta d\mathcal{H}^{m-1}(\xi)\right\} D_\alpha u^k D_\beta u^\ell \\ = 0 \qquad \text{for } j = 1, \dots, n. \\ \text{or} \\ Du(x) = 0. \end{cases} \tag{2.14} $$
Now, let us call a smooth solution of (2.14) *a harmonic map from* $\mathbb{R}^m$ *into a Finsler manifold* (N, F).Compare (2.14) with the standard equation of harmonic maps from $\mathbb{R}^m$ into a Riemannian manifold.

Example. When $(N, F) = (\mathbb{R}^n, F)$ with Minkowskian structure F (i.e. $F(u, X)$ does not depend on u), the identity map from $\mathbb{R}^n$ to $(\mathbb{R}^n, F)$ is a harmonic map. Remark that Mo [15] showed harmonicity of identity map from a Minkowskian space to $\mathbb{R}^n$.

In the following we discuss partial regularity of energy minimizing maps from a Euclidean space $\mathbb{R}^m$ into a Finsler space $(N, F) = (\mathbb{R}^n, F)$.

3 Partial $C^{0,\alpha}$-regularity

In the sequel, we always suppose that $(h_{ij}(u, X))$ satisfies the following conditions:

(h-1) For some concave increasing function ω with $\omega(0) = 0$ we have

$$|h_{ij}(u,X) - h_{ij}(v,X)| \le \omega(|u-v|) \tag{3.1}$$

for all $u, v, X \in \mathbb{R}^n$.

(h-2) There exists a positive constant λ_0 such that

$$h_{ij}(u,X)\xi^i\xi^j = \frac{\partial^2 f(u,X)}{\partial X^i \partial X^j}\xi^i\xi^j > \lambda_0 \|\xi\|^2 \tag{3.2}$$

for all $u, X, \xi \in \mathbb{R}^n$.

Under these assumptions, by the result due to M.Giaquinta and E.Giusti [9, Theorem 5.1], it is known that every minimizer of (1.5) has Hölder-continuous first derivatives on some open subset $\Omega_0 \subset \Omega$ such that $\mathcal{L}^m(\Omega \setminus \Omega_0) = 0$, where $\mathcal{L}^m$ denotes the m-dimensional Lebesgue measure. In contrast, in [10], they also proved that if the target manifold is Riemannian the $(m-3)$-dimensional Hausdorff measure of the singular set of every minimizer is equal to 0. We also mention that, as V.Šverák and X.Yan [16] showed, even if the functional depends only on Du and $m = 3$, we can not expect everywhere $C^{1,\alpha}$-regularity in general.

In this section, we show the following partial $C^{0,\alpha}$-regularity result for the case that $m \le 4$.

Theorem 3.1 ([17]). *Assume that $m \le 4$ and that $h_{ij}(u,X)$ satisfies (3.1) and (3.2). Then a minimizer u of $E(u;\Omega)$ is Hölder continuous on an open subset $\Omega_0 \subset \Omega$ such that $\mathcal{H}^{m-2-\varepsilon}(\Omega \setminus \Omega_0) = 0$.*

Proof. Let $x_0 \in \Omega$ be a arbitrarily fixed point and $R > 0$ a fixed number such that $B_R = B_R(x_0) \subset B_{2R}(x_0) \subset\subset \Omega$. Moreover, in the following we use the following notation:

$$u_R = \frac{1}{|B_R|}\int_{B_R} u\,dx, \tag{3.3}$$

$$du_R = \left(\left(\frac{\partial u^i}{\partial x^\alpha}\right)_R\right) = \left(\frac{1}{|B_R|}\int_{B_R}\frac{\partial u^i}{\partial x^\alpha}dx\right), \tag{3.4}$$

$$a_{ij}^{\alpha\beta}(u,P) = \int_{S^{m-1}} h_{ij}(u, P\cdot\xi)\xi^\alpha\xi^\beta d\mathcal{H}^{m-1}(\xi). \tag{3.5}$$

Let $v \in H^{1,2}(B,\mathbb{R}^n)$ be a solution of the minimizing problem:

$$\begin{cases} E_0(v) := \displaystyle\int_B a_{ij}^{\alpha\beta}(u_R, dv) D_\alpha v^i D_\beta v^j dx \to \min., \\ v - u \in H_0^{1,2}(B,\mathbb{R}^n). \end{cases} \tag{3.6}$$

To get $H^{2,2}$-regularity of a minimizer v, we consider the following equation for the difference quotient

$$\int_{B_R} \{A_{P^i_\alpha}(Dv(x+he_\gamma)) - A_{P^i_\alpha}(Dv(x)\} D_\alpha \varphi^j dx = 0 \quad {}^\forall \varphi \in H^{1,2}_0(\Omega),$$

where

$$A(P) = a^{\alpha\beta}_{ij}(u_R, P)P^i_\alpha P^j_\beta = \int_{S^{m-1}} F^2(u_R, P\xi) d\mathcal{H}^{m-1}(\xi),$$
$$A_{P^i_\alpha}(P) = \frac{\partial A(P)}{\partial P^i_\alpha}.$$

Since $A_{P^i_\alpha}$ are Lipschitz continuous, we can use the chain rule and get

$$\int_\Omega \left\{ \int_0^1 \hat{A}_{P^i_\alpha P^j_\beta}((1-t)Du(x) + tDu(x+he^\gamma))dt \right\} \tau^h_\gamma D_\beta u^j D_\alpha \varphi^i dx = 0,$$

where $\tau^h_\gamma \psi = (\psi(x+he^\gamma) - \psi(x))/h$ and

$$\hat{A}_{P^i_\alpha P^j_\beta}(P) = \begin{cases} A_{P^i_\alpha P^j_\beta}(P) & \text{if} \quad P \neq 0, \\ 0 & \text{if} \quad P = 0. \end{cases}$$

Note that $(1-t)Du(x) + tDu(x+he^\gamma) = 0$ possibly at one value of t except that $Du(x) = Du(x+he^\gamma) = 0$. This implies that, if $\tau^h_\gamma Du \neq 0$,

$$(\mathcal{A}^{\alpha\beta}_{ij}) := \left(\int_0^1 \hat{A}_{P^i_\alpha P^j_\beta}((1-t)Du(x) + tDu(x+he^\gamma))dt \right)$$

is coercive. Namely

$$\mathcal{A}^{\alpha\beta}_{ij} \xi^i_\alpha \xi^j_\beta \geq \nu \|\xi\|^2 \quad {}^\forall \xi \in \mathbb{R}^{mn}, \quad \text{for some } \nu > 0.$$

Thus we can get a Caccioppoli-type estimate and proceed as in the usual cases to get $H^{2,2}$-regularity of v (see for example pp.33–34 in [7]). Now, since v is in the class $H^{2,2}$, we can see that Dv is also a weak solution of an elliptic equation. Therefore, Caccioppoli's inequality holds not only for v but also for Dv, namely for any $r \in (0, R]$ we have

$$\int_{B_{\frac{r}{2}}} |Dv|^2 dx \leq \frac{c_0}{R^2} \int_{B_r} |v - \lambda_1|^2 dx \quad \text{for any } \lambda_1 \in \mathbb{R}^n, \tag{3.7}$$

$$\int_{B_{\frac{r}{2}}} |D^2 v|^2 dx \leq \frac{c_1}{R^2} \int_{B_r} |Dv - \lambda_2|^2 dx \quad \text{for any } \lambda_2 \in \mathbb{R}^{mn}. \tag{3.8}$$

Combining above inequalities for $r = R$ with Sobolev's inequality and writing $2_* = 2m/(m+2)$ (i.e. $(2_*)^* = 2$), we get

$$\int_{B_{\frac{R}{2}}} |Dv|^2 dx \le \frac{c_2}{R^2} \int_{B_R} |v - v_R|^2 dx \le \frac{c_3}{R^2} \left\{ \int_{B_R} |D^2 v|^{2_*} dx \right\}^{2/2_*}$$

$$\int_{B_{\frac{R}{2}}} |D^2 v|^2 dx \le \frac{c_4}{R^2} \int_{B_R} |Dv - (Dv)_R|^2 dx \le \frac{c_5}{R^2} \left\{ \int_{B_R} |D^2 v|^{2_*} dx \right\}^{2/2_*}$$

or equivalently

$$\left\{ \fint_{B_{\frac{R}{2}}} |Dv|^2 dx \right\}^{\frac{1}{2}} \le c_6 \left\{ \fint_{B_R} |Dv|^{2_*} dx \right\}^{1/2_*}, \tag{3.9}$$

$$\left\{ \fint_{B_{\frac{R}{2}}} |D^2 v|^2 dx \right\}^{\frac{1}{2}} \le c_7 \left\{ \fint_{B_R} |D^2 v|^{2_*} dx \right\}^{1/2_*}. \tag{3.10}$$

Since $2_* < 2$, using the local version of Gehring's inequality due to M.Giaquinta and G.Modica [11, Proposition 5.1], we obtain from (3.9) and (3.10) the reverse Hölder inequalities for Dv and $D^2 v$:

$$\left\{ \fint_{B_{\frac{R}{2}}} |Dv|^p dx \right\}^{\frac{1}{p}} \le c_8 \left\{ \fint_{B_R} |Dv|^2 dx \right\}^{1/2}, \tag{3.11}$$

$$\left\{ \fint_{B_{\frac{R}{2}}} |D^2 v|^p dx \right\}^{\frac{1}{p}} \le c_9 \left\{ \fint_{B_R} |D^2 v|^2 dx \right\}^{1/2} \tag{3.12}$$

for some $p > 2$.

Let η be a function of the class $C_0^\infty(B_{R/2})$ which satisfies $1 \ge \eta \ge 0$, $\eta = 1$ in $B_{R/4}$ and $|D\eta| \le c_{10}(m)/R$. Then, applying Sobolev's inequality to ηDv, we get

$$\begin{aligned} \|\eta Dv\|_{L^{p^*}(B_{R/2})} &\le c_{11}(m) \|D(\eta Dv)\|_{L^p(B_{R/2})} \\ &\le \frac{c_{12}}{R} \|Dv\|_{L^p(B_{R/2})} + c_{12} \|D^2 v\|_{L^p(B_{R/2})}. \end{aligned} \tag{3.13}$$

Combining (3.13) with (3.11), (3.12) and (3.8), we obtain

$$\|Dv\|_{L^{p^*}(B_{R/4})} \le c_{13} R^{-1-\frac{m}{2}+\frac{m}{p}} \|Dv\|_{L^2(B_R)}, \tag{3.14}$$

where c_{13} does not depend on v and R.

Now, put $\delta = 1 - \frac{m}{p^*} = 2 - \frac{m}{p}$. Since $p > 2$, $\delta > 0$ for $m \le 4$. Using (3.14) and Morrey-type estimate, we get for $\rho < R/4$

$$\begin{aligned} & \left\{ \rho^{-m-2\delta} \int_{B_\rho} |v - v_\rho|^2 dx \right\}^{1/2} \\ \le\ & \sup_{B_{R/4}} \frac{|v(x) - v(y)|}{|x-y|^\delta} \\ \le\ & c_{14}(1 + R^\delta)\|Dv\|_{L^{p^*}(B_{R/4})} \quad \text{(cf. (7.43) in [12].)} \\ \le\ & c_{15} R^{-1-\frac{m}{2}+\frac{m}{p}} \|Dv\|_{L^2(B_R)} \\ =\ & c_{16} R^{1-\frac{m}{2}-\delta} \|Dv\|_{L^2(B_R)}. \end{aligned} \tag{3.15}$$

On the other hand, from (3.7) we have

$$\rho^{-m-2\delta+2} \int_{B_{\rho/2}} |Dv|^2 dx \le c_{17} \rho^{-m-2\delta} \int_{B_\rho} |v - v_\rho|^2 dx. \tag{3.16}$$

Combining (3.15) and (3.16), we get

$$\rho^{-m+2-2\delta} \int_{B_\rho} |Dv|^2 dx \le c_{18} R^{-m+2-2\delta} \int_{B_R} |Dv|^2 dx \tag{3.17}$$

for all $\rho < R/8$.

Let $w = u - v$. We can estimate $\int |Dw|^2 dx$ proceeding as in [9] to see

$$\begin{aligned} & \int_{B_R} |Dw|^2 dx \\ \le\ & c_{19} \left\{ \omega\left(c_{20} R^{2-m} \int_{B_R} |Du|^2 dx\right) \right\}^{1-\frac{2}{p}} \int_{B_{2R}} |Du|^2 dx. \end{aligned} \tag{3.18}$$

Using (3.17) and (3.18), we obtain the following estimate:

$$\begin{aligned} & \int_{B_\rho(x_0)} |Du|^2 dx \\ \le\ & \int_{B_\rho(x_0)} |Dv|^2 dx + \int_{B_\rho(x_0)} |Dw|^2 dx \\ \le\ & c_{21} \left(\left(\frac{\rho}{R}\right)^{m-2+2\delta} + \omega\left(C R^{2-m} \int_{B_R(x_0)} |Du|^2 dx \right)^{1-\frac{2}{p}} \right) \\ & \times \int_{B_{2R}(x_0)} |Du|^2 dx. \end{aligned} \tag{3.19}$$

Now, we can use [9, Lemma 2.2] to get partial $C^{0.\alpha}$-regularity of u. Namely, for any $\alpha \in (0, \delta)$, by virtue of [9, Lemma 2.2], we can take $\varepsilon > 0$ sufficiently small so that

$$\rho^{-m+2-2\alpha} \int_{B_\rho(x_0)} |Du|^2 dx < c_{22}$$

for any x_0 in

$$\Omega_0 = \{x \in \Omega; \liminf_{R\to 0} R^{-m+2} \int_{B_R(x)} |Du|^2 dx < \varepsilon\}.$$

This implies that $u \in C^{0,\alpha}(\Omega_0)$ and that $\mathcal{H}^{m-2-\varepsilon}(\Omega\backslash\Omega_0) = 0$. (See, for example, [7, p.43] and [7, Theorem 6.2]). $\square$

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